Dynamic Behavior and Unstable State Evolution of Ocean-Atmosphere Oscillator *

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ABSTRACT

It is mathematically and thoroughly proved in this paper that the nonlinear stochastic ocean-atmosphere oscillator model possesses a stable limit cycle; then the model equations are transformed into the Fokker-Planck equation (FPE), and the evolution of El Niño-Southern Oscillation (ENSO) from unstable state to stable state is studied from the point of view of nonequilibrium system dynamics. The study results reveal that although the complex nonlinear ocean-atmosphere oscillator model possesses multiequilibrium states, the real climatic system possesses only a quasi-normal state and a strong ENSO cycle stable state. The first passage time between states is also given in this paper, and the theoretical computational results agree with observational data.

Key words: ENSO, Fokker-Planck equation, limit cycle, ocean-atmosphere oscillator, probability density

1. Introduction

The Southern Oscillation and El Niño/La Niña are anomalous events which occur in the tropical atmosphere and ocean respectively, and their physical mechanisms have been successively studied in the recent half century. Bjerknes (1969) pointed out that the close link between Walker circulation and SST in the equatorial Pacific is the manifestation of the interaction of tropical atmospheric and oceanic motions. Therefore, Southern Oscillations and El Niño cycles were described as a unified climatic phenomenon and abbreviated as ENSO. El Niño and La Niña are complementary to each other. A warm phase, or El Niño and a cold phase, or La Niña consist of a cycle.

Since Bjerknes presented the concept of ENSO, this strong signal occurring in the equatorial Pacific ocean-atmosphere system has been amply investigated. On the one hand, ENSO has attracted worldwide attention (Barber and Chavez, 1983; Ropelewski and Halpert, 1987), for it affects regional and global climate, especially severe disasters of droughts/floods. Previous studies indicated that El Niño and La Niña play an important role. On the other hand, ENSO also possesses an important influence on China climate, especially on the East Asian monsoon, the summer cooling damage in Northeast China, droughts/floods in the middle-lower reaches of the Changjiang River in the monsoon area, and so on. The above facts indicate that China climate as well as global climate contains prominent ENSO signals (Wu and Meng, 1998; Li and Mu, 1998; Jin and Tao, 1999; Li, 1988). And therefore revealing the possible cycle and triggering mechanism of ENSO is of great importance for understanding and predicting ENSO.

The complexity and nonlinearity of ocean-atmosphere interaction lead to a great difficulty in the theoretical explanation of its mechanism by using simple models, and in consequence the numerical simulation becomes the major method for revealing the temporal and spatial structure of ENSO. The ENSO cycle possesses periodicity, scale-selecting character, and anomalous physical structure; therefore it is only from the nonlinear dynamic climatology that the further understanding of ENSO cycles can be obtained. The complicated ocean-atmosphere coupled model can simulate the life cycle of ENSO and the nonperiodic interannual change rate. This is no

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doubt an important progress in understanding the ENSO phenomenon. Nevertheless viewed from the thoroughly understanding of the thermal/dynamic mechanisms of ENSO, some physical processes remain to be further clarified and thoroughly studied. Among many problems that need to be explored, emphasis should be placed on several key problems, e.g., how does the life cycle of ENSO complete? How is a warm El Niño state transformed into a cold La Niña state? How does the nonperiodic ENSO cycle generate? These problems remain to be solved by the application of the latest achievements of nonlinear science in the study of ENSO.

Mantua and Battisti (1995) considered that the periodicity of ENSO might arise from high frequency stochastic forcing or the intrinsic nonlinearity of the ENSO system. Mesoscale coupled model experiments suggested that the nonlinearity of ENSO is essentially an evolitional behavior of the low-dimension chaos driven by the annual cycle of basic state. On the above basis and through physical simplification and mathematical deduction, Wang and Fang (1996; 1999) established the stochastic dynamic model of sea-air oscillator for ENSO events, which possesses a unique limit cycle solution (a stable attractor) representing an intrinsic interannual oscillator of the coupled system.

Wang’s nonlinear stochastic ocean-atmosphere oscillator model is adopted in this paper, and it is mathematically and thoroughly proved that the nonlinear stochastic ocean-atmosphere oscillator model possesses a stable limit cycle; then the model equations are transformed into the FPE (Fokker-Planck equation), and the evolution of ENSO from unstable state to stable state is studied for the first time from the angle of nonequilibrium system dynamics. The study results reveal that although the complicated nonlinear ocean-atmosphere oscillator model possesses multiequilibrium states, the real climatic system possesses only a quasi-normal state and a strong ENSO cycle stable state. The first passage time between states is also given, and the theoretical computational results agree with the observational data.

2. Stochastic dynamic model for ENSO events

Stochastic dynamic equations for ENSO events (Wang and Fang, 1999), which govern the anomalous SST, T, and the thermocline depth, h, are

\[ \frac{\partial T}{\partial t} = a_1 T + a_2 h + a_3 T^2 - a_3 \mu' T h - 2T^3 + \Gamma(t), \]

\[ \frac{\partial h}{\partial t} = 2bh - bT - 2h^3 + \Gamma(t), \]

where parameters \(a_1, a_2, a_3, \mu',\) and \(b\) can be found in Wang and Fang (1999), and \(\Gamma(t)\) is the internal or external stochastic noise of the ENSO system. The deterministic equations are rescaled as follows. Let

\[ x = \sqrt{2}T - \frac{a_3}{3\sqrt{2}}, \quad y = \sqrt{2}h, \]

and then substituting them into Eq.(1) yields

\[ \frac{dx}{dt} = a_1 x - x^3 - \xi_1 (x - y) + \mu xy + \Gamma(t), \]

\[ \frac{dy}{dt} = a_2 y - y^3 - \xi_2 (y - x) + \Gamma(t), \]

(2)

where

\[ a_1 = a_1 + a_2 + a_3^2 (1 - \mu')/6, \quad \xi_1 = a_2 - a_3^2 \mu'/6, \]

\[ \mu = -\frac{\sqrt{2}}{2} a_3 \mu', \quad \alpha_2 = b, \quad \xi_2 = -b. \]

Adjusting the parameters so that \(a_1 = a_2 = \alpha, \quad \xi_1 = \xi_2 = \varepsilon,\) then Eq.(2) becomes

\[ \frac{dx}{dt} = \alpha x - x^3 - \varepsilon (x - y) + \mu xy + \Gamma(t), \]

\[ \frac{dy}{dt} = \alpha y - y^3 - \varepsilon (y - x) + \Gamma(t). \]

(3)

\(\Gamma(t)\) is generally assumed to be the white noise, and \(< \Gamma(t) >= 0, < \Gamma(t') \Gamma(t) >= 2D(t - t'),\) and \(D\) is the intensity of noise. By drawing an analogy to Schlogl chemical reaction model in chemistry, \(\xi\) can be
considered as the diffusion between $T$ and $h$, $\mu xy$ as a cross term, and also $\mu$ is a minor quantity, and $\varepsilon > 0$. Equation (3) is a system of stochastic nonlinear equations describing the complicated nonlinear effect between anomalous SST, $T$, and thermocline depth, $h$.

3. Steady state solutions of the ocean-atmosphere dynamic system

Because it is difficult to analytically solve a complicated system of nonlinear equations regardless of differential equations or stochastic dynamic equations, we first get the stable state solution of the system, and then further explore the evolution of the solution in the neighborhood of the stable state with instability analysis or linearization.

After neglecting the cross term, the deterministic equations describing ENSO events are

$$\frac{dx}{dt} = ax - x^3 - \varepsilon(x - y),$$

$$\frac{dy}{dt} = ay - y^3 - \varepsilon(y - x).$$

Equation (4) is the Schlögl double-box model in chemistry. In dynamical climatologic box models, the atmosphere and ocean are respectively taken as a box, on which the radiation-convection model is established. The derivation of Eq.(4) in this paper not only demonstrates the rationality of box model but also proves that some conceptual theories are universal in different disciplines.

3.1 A stable limit cycle

We are first going to prove that Eq.(4) possesses a unique stable limit cycle, which describes the quasi-normal steady state. The limit cycle is a stable attractor representing the inherent interannual oscillations of the coupled system. After complex deduction, Eq. (4) may be rewritten into a higher order equation

$$\frac{d^2y}{dt^2} = B_1(y)\frac{dy}{dt} + B_2(y)\frac{dy}{dt}^2 + B_3(y)\frac{dy}{dt} + B_4(y),$$

where

$$B_1(y) = -\frac{1}{\varepsilon^3}, \quad B_2(y) = \frac{3}{\varepsilon^3}(Ay - y^3),$$

$$B_3(y) = 2A - 3(1 + \frac{1}{\varepsilon^3}A^2)y^2 + \frac{6A}{\varepsilon^3}y^4 - \frac{3}{\varepsilon^3}y^6;$$

$$B_4(y) = \frac{1}{\varepsilon^3}y^9 + \frac{3A}{\varepsilon^3}y^7 - \frac{3A^2}{\varepsilon^3}y^5 + \frac{(A^2 + A)y^3}{\varepsilon^3} + (\varepsilon - A^2)y,$$

$$A = \alpha - \varepsilon.$$

Set $\frac{dy}{dt} = z$, then

$$\dot{z} = B_1(y)(z)^3 + B_2(y)(z)^2 + B_3(y)(z) + B_4(y).$$

Generally, as far as the limit cycle is concerned, the total energy of a system on the orbit of a limit cycle does not change. Assume that the total energy of the system is $E$,

$$E = E_D + E_P = \frac{1}{2}\dot{y}^2 + \frac{1}{4}y^2,$$

where $E_D$ and $E_P$ represent dynamic and potential energy, respectively. Therefore,

$$\int_{t}^{t+T} \frac{dE}{dt} dt = 0,$$

i.e.,

$$\int_{t}^{t+T} \frac{d}{dt}(\frac{1}{2}\dot{y}^2 + \frac{1}{2}y^2) dt = \int_{t}^{t+T} (\dot{y}\ddot{y} + \ddot{y}y) dt = 0. \quad (8)$$

Known from Eq.(6),

$$\int_{t}^{t+T} \left\{B_1(y)\dot{y}^4 + B_2(y)\dot{y}^3 + B_3(y)\dot{y}^2 + [y + B_4(y)]\dot{y}\right\} dt = 0. \quad (9)$$

Assume that the mean radius of the limit cycle is $\bar{r}$, and then the periodic solution for the system is

$$y = \bar{r}\cos t, \quad \dot{y} = -\bar{r}\sin t. \quad (10)$$

And let $t = 0, T = 2\pi$, then

$$\int_{0}^{2\pi} B_1(y)\dot{y}^4 dt = \frac{1}{\varepsilon^3}r^4\left(-\frac{1}{\varepsilon^3}\sin3tcost\right)_{0}^{2\pi}$$

$$= \frac{3}{4\varepsilon^3}r^4(\frac{1}{2} - \frac{1}{4}\sin2t)_{0}^{2\pi}$$

$$= \frac{3}{4\varepsilon^3}r^4\pi, \quad (11)$$
\[
\int_0^{2\pi} y^n y^3 dt = -\int_0^{2\pi} \frac{r^{n+3} \cos^n t \sin^3 t dt}{r^{n+3}} = 0
\]
i.e.,
\[
\int_0^{2\pi} B_2(y) y^3 dt = 0. \tag{12}
\]
Similarly,
\[
\int_0^{2\pi} y^n y dt = 0, \quad \int_0^{2\pi} B_4(y) y dt = 0. \tag{13}
\]
In addition,
\[
\int_0^{2\pi} y^2 dt = \bar{r} \pi, \quad \int_0^{2\pi} y^2 dt = \frac{1}{4} \bar{r}^4 \pi,
\]
\[
\int_0^{2\pi} y^4 y^2 dt = \frac{1}{8} \bar{r}^6 \pi, \quad \int_0^{2\pi} y^5 y^2 dt = \frac{5}{64} \bar{r}^8 \pi,
\]
\[
\int_0^{2\pi} y^{2n+1} y^2 dt = 0. \tag{14}
\]
Substituting Eqs.(11)-(14) into Eq.(9) yields
\[
\int_{t}^{t+T} \left\{ B_1(y) y^4 + B_2(y) y^3 + B_3(y) y^2 + |y + B_4(y)| y \right\} dt = \bar{r}^2 \pi \left[ -\frac{15}{64 \epsilon^3} \bar{r}^6 + \frac{3A}{4 \epsilon^3} \bar{r}^4 - \frac{3}{4} \bar{r}^2 \right.
\]
\[
\left. -\frac{3A^2}{4 \epsilon^3} \bar{r}^2 - \frac{3}{4 \epsilon^3} \bar{r}^2 + 2A \right] = 0. \tag{15}
\]
Set \( \bar{r}^2 = f \), then
\[
-\frac{15}{64 \epsilon^3} f \pi [f^3 - \frac{16}{5} Af^2 + \frac{16}{5} (\epsilon^3 + A^2 + 1) f - \frac{128}{15} \epsilon \epsilon^3] = 0. \tag{16}
\]
Suppose \( f' = f + \frac{16A}{15} \), then the second factor in the left hand side of Eq.(16) becomes
\[
f'^3 + pf + q = 0, \tag{17}
\]
where \( p, q \) can be expressed in terms of \( A, \epsilon \). The three roots of Eq.(17) are respectively
\[
f'_1 = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}},
\]
\[
f'_2 = \omega_1 \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \omega_2 \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}},
\]
\[
f'_3 = \omega_2 \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \omega_1 \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \tag{18}
\]
where \( \omega_1 = -1 + \sqrt{3}i \), \( \omega_2 = -1 - \sqrt{3}i \). Through calculation, when \( 3 \epsilon^3 > 1 \), \( f'_1 \) is a positive real root and \( \bar{r}^2 = B(B = -\frac{16A}{15} + f'_1) \). When \( \bar{r}^2 < B \), \( f'_1 T \frac{dE}{dt} > 0 \); and when \( \bar{r}^2 > B \), \( f'_1 T \frac{dE}{dt} < 0 \). The above operations sufficiently indicate that as time goes on the movement inside of the limit cycle will approximate to it due to the increase of energy, and meanwhile the movement outside of the limit cycle will approximate to it due to the decrease of energy, thus prove that there is a stable limit cycle in the ENSO event dynamic system, which suggests that other factors excite ENSO. However for the reason of energy conservation, energy is adjusted and redistributed in the global range, and the turnover of El Niño or La Niña may result from energy conservation. In fact, the atmosphere and oceans are dynamically and thermodynamically adjusting each other, thus once the anomalous condition is so strong that the equilibrium state between them is disrupted, then the adjustment under new conditions may make anomalous processes possible to sustain and to develop into El Niño or La Niña quasi-equilibrium states.

3.2 Steady state solution

Set \( \frac{dx}{dt} = 0 \), \( \frac{dy}{dt} = 0 \), then
\[
\alpha x - x^3 - \epsilon (x - y) = 0, \tag{19a}
\]
\[
\alpha y - y^3 - \epsilon (y - x) = 0. \tag{19b}
\]
When \( \alpha < 0 \), Eq.(19) has only a steady state solution \( x_0 = 0 \), \( y_0 = 0 \), which can be considered as a quasi-normal steady state of air-sea interaction. If the air-sea interaction maintains under the condition of \( \alpha < 0 \), the prediction and observation of climate will
be more simplified. In fact, under the influence of external noises, the internal stochastic fluctuations of the atmosphere-ocean coupled system may enable \( \alpha > 0 \), thus resulting in the anomalous climate which is complicated and difficult to predict.

When \( \alpha > 0 \) and \( \alpha > 3\epsilon \), the solution of \( x_0 = 0, y_0 = 0 \) becomes an unstable saddle point, and the equation system (19) has a pair of symmetric steady nodal points \((x_1 = y_1 = \sqrt{\alpha}, x_2 = y_2 = -\sqrt{\alpha})\) and a pair of antisymmetric steady nodal points \((x_3 = y_3 = \sqrt{\alpha - 2\epsilon}, x_4 = y_4 = -\sqrt{\alpha - 2\epsilon})\), as well as four periodic solutions. Obviously, \( \alpha = 0 \) is a bifurcation point.

Given a deterministic system such as Eq. (19), there are two categories of steady state solutions: steady states and unstable states, which have completely different asymptotic behavior. When the system is near a steady state, it will approach the state as time goes on; but when the system is near an unstable state, it will depart from the state. Therefore it is seemingly that there are no unstable states in reality and there is no real meaning in the study on the evolution of unstable state. However the real circumstances are not the case, the evolution of unstable states happens in many processes of real meaning in nonlinear systems, therefore studies on the characters of the evolution and on the effect of stochastic forcing on the evolution are important issues of stochastic theory for nonlinear systems.

4. Evolution of the unstable state of atmosphere-ocean coupled oscillator

4.1 ENSO events and the Fokker-Plank equation

If change parameters such that \( \alpha < 0 \) suddenly becomes \( \alpha > 0 \), the system values \((x, y)\) still remain in the vicinity of the original stable solution \((x_0 = 0, y_0 = 0\), noticeable, the solution now becomes unstable).

Equation (3) describing the atmosphere-ocean coupled oscillator of the anomalous SST, \( T \), and the thermocline depth, \( h \), may be transformed into an FPE (Risken, 1983; Hu and Lu, 1992)

\[
\frac{\partial p(x, y, t)}{\partial t} = -\frac{\partial}{\partial x}[c_1(x, y)]\rho(x, y, t) - \frac{\partial}{\partial y}[c_2(x, y)]\rho(x, y, t) + D(\partial^2 x^2 + \partial^2 y^2)\rho(x, y, t),
\]

where \( c_1(x, y) = \alpha x - x^3 - \epsilon (x - y), c_2(x, y) = \alpha y - y^3 - \epsilon (y - x) \), and \( D \) is the intensity of noises. Thus variables are described by probability density \( \rho(x, y, t) \), and their measurement values are given by statistical means, i.e.,

\[
\langle x(t) \rangle = \int x\rho(x, y, t)dx\,dy,
\]

\[
\langle y(t) \rangle = \int y\rho(x, y, t)dx\,dy,
\]

then the steady state solution of Eq.(20) may be written as

\[
\rho(x, y) = N\exp\left(-\frac{\phi(x, y)}{D}\right),
\]

where the generalized potential function

\[
\phi(x, y) = -\frac{\alpha}{2}x^2 - \frac{1}{4}x^4 + \frac{\alpha}{2}y^2 - \frac{1}{4}y^4 - 2\epsilon(x - y)^2,
\]

\( \alpha \) is a constant, and determined by the normalizing condition \( N = \left[ \int_{-\infty}^{\infty} \exp\left(-\frac{\phi(x, y)}{D}\right)dx\,dy \right]^{-1} \). For different \( \alpha \) and \( \epsilon \), shapes of the generalized potential function differ greatly, and it of course has different properties. Figure 1a displays that when \( \alpha < 0 \), there exists only one steady state; when \( \alpha > 0 \), if \( \epsilon \) takes different values, the generalized potential function exhibits complex shape figures (Figs.1b and 1c). In this paper only the potential function when \( \alpha > 3\epsilon \) is studied, and it has four minimum values \((x_i, y_i; i = 1, 2, 3, \text{and} 4)\), a maximum value \((0, 0)\) and four saddle points. Among them the four minimum values correspond to the four sets of steady state solutions for the deterministic Eq.(19). In order to describe the evolution of unstable states of the ENSO system, the \( \Omega \) expansion of two-dimensional Green function (\( \Omega \) FPE) is employed in this paper.

For the relaxation process from an unstable state to a quasi-stable state, suppose the initial distribution is

\[
\rho(x, y; 0) = \delta(x - \gamma\sqrt{D})\delta(y - \beta\sqrt{D}),
\]

\( \gamma, \beta = O(1) \).

The solution, \( \rho_{in}(u, v, t) \), of the FPE of Eq.(20) linearized in the vicinity of \((0, 0)\) (Hu and Lu, 1992).
is
\[ \rho_{\infty}(u, v, t) = \frac{1}{\pi D} \sqrt{\frac{\alpha(\alpha - 2\varepsilon)}{(1 - e^{2\alpha t})(1 - e^{2(\alpha - 2\varepsilon)t})}} \exp\left[ \frac{\alpha}{D} \frac{(u - u_0 e^{2\alpha t})^2}{1 - e^{2\alpha t}} + \frac{\alpha - 2\varepsilon}{D} \frac{(v - v_0 e^{(\alpha - 2\varepsilon)t})^2}{1 - e^{2(\alpha - 2\varepsilon)t}} \right] \]
where
\[ u = \frac{\sqrt{2}}{2} (x + y), \quad v = \frac{\sqrt{2}}{2} (x - y), \]
\[ u_0 = \frac{\sqrt{2}}{2} (\gamma + \beta) \sqrt{D}, \quad v_0 = \frac{\sqrt{2}}{2} (\gamma - \beta) \sqrt{D}. \]

As long as \( e^{2\alpha t} \ll D^{-1} \), Eq.(24) is a good description of the temporal evolution process for Eq.(20); when \( e^{2\alpha t} \gg 1 \), most of probability has overflowed the unstable region \( |x|, |y| \leq \sqrt{D}. \) Due to \( D \ll 1 \), it is easy to find \( t_s \), satisfying
\[ \frac{1}{D} \gg e^{2\alpha t_s} \gg 1. \] (25)

At this very moment, the linear approximate solution, Eq.(20), of drift forcing is still valid, and the \( \Omega \) FPE is applicable.

For the Delta distribution of \( \rho(x_s, y_s, t_s) = \delta(x - x_s) \delta(y - y_s) \) at \( t_s \) time, its \( \Omega \) expansion approximate solution is
\[ \rho_{\Omega}(x, y, t) = \frac{1}{\sqrt{\pi D} \det \sigma} \exp\left[ - \frac{1}{D} (H' - H_0'(t)) \sigma^{-1} (H - H_0(t)) \right], \] (26)

Fig.1. Generalized potential function diagrams under the condition of different parameters. (a) \( \alpha = -0.05, \varepsilon = 0.50 \); (b) \( \alpha = 0.05, \varepsilon = 0.01 \); and (c) \( \alpha = 0.01, \varepsilon = 0.50 \).
where $H$ and $H_0(t)$ are two-dimensional vectors, 
$H = \begin{pmatrix} x \\ y \end{pmatrix}$, $H_0(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, and $H'$ and $H'_0(t)$ are 
their transpositions respectively; $\sigma$ is a $2 \times 2$ matrix 
$\sigma = \begin{pmatrix} \sigma_{xx}(t) & \sigma_{xy}(t) \\ \sigma_{xy}(t) & \sigma_{yy}(t) \end{pmatrix}$; $x(t)$ and $y(t)$ are determined 
by deterministic Eq.(4), and their initial conditions 
are $x(t_s) = x_s$, $y(t_s) = y_s$, respectively. Elements of 
matrix $\sigma$ are given in the following equation system 
\begin{align*}
\dot{\sigma}_{xx} &= 2[(\alpha - 3) - 3x(t)^2]\sigma_{xx} + 2 + \varepsilon\sigma_{xy}, \\
\dot{\sigma}_{xy} &= \varepsilon(\sigma_{xx} + \sigma_{yy}) + \{(\alpha - 3) - 3x(t)^2\} \\
&+ [(\alpha - 3) - 3y(t)^2]\sigma_{xy}, \\
\dot{\sigma}_{yy} &= 2[(\alpha - 3) - 3y(t)^2]\sigma_{yy} + 2 + \varepsilon\sigma_{xy},
\end{align*}
(27)
and their initial conditions are $\sigma_{xx}(t_s) = \sigma_{xy}(t_s) = 
\sigma_{yy}(t_s) = 0$. Connecting Eqs.(24) and (26) yields the 
probability of the evolutionary process from unstable 
state to metastable state 
\begin{align*}
\rho(x, y, t) &= \int \rho_0\left[\frac{\sqrt{\gamma}}{2}(x + y), \frac{\sqrt{\gamma}}{2}(x - y), t\right] \\
&\cdot \rho_1(x, y, t) dx dy ds.
\end{align*}
(28)
Thus, the problem of solving two-dimensional second-order 
partial differential equation system (2) becomes a bivariate 
ordinary differential equation.

4.2 Discussion on the two-dimensional reduction of ENSO events

When $\frac{1}{\tau} \gg e^{2\alpha t} \gg 1$, then 
\begin{equation}
\frac{e^{2\alpha t}}{e^{2(\alpha - 2\varepsilon)t}} = e^{4\alpha t} \gg 1.
\end{equation}
(29)
Therefore, it is as early as in the linear area, the diffusion 
of the probability distribution of Eq.(24) at the 
speed of $e^{\alpha t}$ in the unstable manifold ($\mu$) direction has 
been far greater than that at the speed of $e^{(\alpha - 2\varepsilon)t}$ in 
the secondary unstable manifold ($\nu$) direction. The probability 
distribution possesses a certain width in $\nu$ direction and extends to a certain length in $\mu$ direction. The greater the $t$ becomes, the larger the ratio 
of length to width of the long narrow strip is. When 
t \gg 1, the probability distribution can be considered to be one dimension in $\mu$ direction. When the linear 
area solution is connected with the $\Omega$ expansion solution at $t_s$ time, the latter faces such an initial condition 
that the initial distribution has been greatly extended in $u_s$ direction but is still approximately a Delta distribution in $v_s$ direction. Therefore the deterministic equations can be reduced to 
\begin{align*}
\frac{dx}{dt} &= \alpha x - x^3, \\
y(t) &= x(t).
\end{align*}
(30)
The elements of matrix $\sigma$ can be reduced to 
\begin{align*}
\dot{\sigma}_{xx} &= 2[(\alpha - 3) - 3x(t)^2]\sigma_{xx} + 2 + \varepsilon\sigma_{xy}, \\
\dot{\sigma}_{xy} &= \sigma_{xx}, \\
\dot{\sigma}_{xy} &= 2\varepsilon\sigma_{xx} + 2[(\alpha - 3) - 3x(t)^2]\sigma_{xy}.
\end{align*}
Their solutions can be immediately given as follows 
\begin{align*}
\frac{dx(t)}{\alpha x - x^3} &= t - t_s, \\
y(t) &= x(t), \\
\sigma_{xx} + \sigma_{xy} &= (\alpha x - x^3)^2 \int_{x_s}^{x} \frac{2dx}{(\alpha x - x^3)^2}, \\
\sigma_{xx} - \sigma_{xy} &= [(\alpha - 2\varepsilon)x - x^3]^2 \\
&\cdot \int_{x_s}^{x} \frac{2dx}{[(\alpha - 2\varepsilon)x - x^3]^2}, \\
\sigma_{xx} &= \sigma_{yy}.
\end{align*}
(31)
By using the linear transformations $u = \frac{\sqrt{\gamma}}{2}(x + y)$ and 
v = $\frac{\sqrt{\gamma}}{2}(x - y)$, matrix $\sigma$ can be rewritten in a diagonal form 
\begin{align*}
\sigma_{uu} &= \frac{\sqrt{\gamma}}{2}(\sigma_{xx} + \sigma_{xy}) = Z_1 \int_{x_s}^{x} \frac{2dx}{Z_1}, \\
\sigma_{vv} &= \frac{\sqrt{\gamma}}{2}(\sigma_{xx} - \sigma_{xy}) = Z_2 \int_{x_s}^{x} \frac{2dx}{Z_2}, \\
Z_1 &= \alpha x - x^3, \\
Z_2 &= (\alpha - 2\varepsilon)x - x^3.
\end{align*}
Thus the evolutionary probability of the system from 
unstable state to metastable state is 
\begin{align*}
\rho(u, v, t) &= \frac{1}{4\pi^2 D^2 \sigma_{uu} \sigma_{vv}} \\
&\cdot \int du_s dv_s \exp\left\{-\frac{1}{2D} \left(\frac{(u_s - u_0 e^{\alpha t})^2}{\sigma_{uu}} + \frac{(u - u(t))^2}{\sigma_{uu}} + \frac{v^2}{\sigma_{vv}}\right)\right\}, \\
&\cdot \left\{\frac{(u - u_0 e^{\alpha t})^2}{\sigma_{uu}} + \frac{v^2}{\sigma_{vv}}\right\},
\end{align*}
(33)
which is a Gaussian distribution with respect to $v_s$. Feng (1996) found an exact time-dependent solution of ocean-atmosphere coupled stochastic dynamic model by employing a quantum-mechanical method, and the analysis of the time-dependent solution indicates that the ocean-atmosphere system behaves in a way of Brownian movement if the system is in ground state, thus proving the standing point of Hasselmann’s stochastic climate model (Hasselmann, 1976), i.e., climate change can be considered as a Brownian movement of light and heavy particle collision.

The system starting from different steady states will finally develop into which of the following steady states?

\[
x_1 = y_1 = \sqrt{\alpha}, \quad x_2 = y_2 = -\sqrt{\alpha},
\]

\[
x_3 = y_3 = \sqrt{\alpha - 2\varepsilon}, \quad x_4 = y_4 = -\sqrt{\alpha - 2\varepsilon}. \tag{34}
\]

In fact, it can be answered before the time when integral (28) is solved, because the conclusion has been determined in the modal competition in the vicinity of the unstable point in the initial time area. Equation (24) possesses two modules $u$ and $v$ among them and the unstable module $u$ dominates. It occupies a dominate position in the modal competition near the unstable point, and thus dominates the system so that the problem of two modules in fact becomes a one-dimensional problem centered with the most unstable module. Especially the inclusion of stochastic factors makes the property of the FPE quite different from the corresponding deterministic equations. When $\gamma = \beta = 0$, the deterministic equation cannot determine which steady state the system will develop to, however the conclusion of solution (28) is quite deterministic. When $t \to \infty$, all the probability of $u_s > 0$ moves towards the potential well of $x_1 = y_1 = \sqrt{\alpha}$ (the right side of Fig.2), while the probability of $u_s < 0$ towards that of $x_2 = y_2 = -\sqrt{\alpha}$ (the left side of Fig.2). And the resultant metastable state distribution is

\[
\rho(u, v, t) = \frac{1}{2\pi D} \frac{(2\alpha - \varepsilon)^2}{\sqrt{\alpha(\alpha - \varepsilon)^2 + 2\pi D}} e^{-\frac{\sqrt{2\alpha - \varepsilon}^2(u - \sqrt{\alpha})^2}{2D}}
\]

\[
\times \left[ e^{-\frac{\sqrt{2\alpha - \varepsilon}^2(u - \sqrt{\alpha})^2}{2D\alpha}} + e^{-\frac{\sqrt{2\alpha - \varepsilon}^2(u + \sqrt{\alpha})^2}{2D\alpha}} \right], \tag{35}
\]

i.e., the overwhelming majority probability concentrates half-and-half in the two potential wells of symmetric solutions $(x_1, y_1)$ and $(x_2, y_2)$, while there is almost no probability distribution in the potential wells of two antisymmetric solutions $(x_3, y_3)$ and $(x_4, y_4)$. It is especially interesting that if the initial Delta distribution is at the point of $x_0 + y_0 = u_0\sqrt{D} = 0$, $x_0 - y_0 = v_0\sqrt{D} \neq 0$ (for example $v_0 > 0$) instead of at the origin, as known from the deterministic equations, the system must finally develop to $(x_3, y_3)$; however under the effect of stochastic forcing, when $t \to \infty$, the overwhelming majority probability still distributes half-and-half in the first two symmetric potential wells, and there is almost no probability distribution in the potential wells of $(x_3, y_3)$ and $(x_4, y_4)$. This idea is quite similar to the dominant principle in synergetic theory, and the difference between them is that synergetics emphasizes the dominance of the unstable slow module over all stable and secondary unstable modules, thus determining the destiny of the system and producing coordination effects. Besides, handling the whole process from the unstable state to the steady state with the $\Omega$ FPE theory permits that control parameters may be far greater than threshold values, but the dominant principle in synergetic

![Fig.2. The first passage time for ENSO events under different $\alpha$ and $D$ conditions (the abscissa denotes the intensity of noises, $D$, and the ordinate $\alpha$).](image-url)
theory concentrates on discussing smaller variations of parameters close to threshold values.

Generally speaking, along with $t \to \infty$, the final state that the initial unstable system can arrive at has to be the stable state pointed by the unstable manifold. The antisymmetric steady state in the really evolitional process from the most unstable state to the steady state can not be observed, as if the system possesses only two steady states, and their probability is almost one half.

In fact, there exists transition, i.e., probability exchange, between the two symmetric steady states, and here only computational formulae for the lifetime of system in two symmetric steady states, i.e., the first passage time, are given as follows:

$$T_1 = \frac{1}{2\pi} \sqrt{\phi''(x_1)\phi''(x_0)} \exp\left(\frac{\phi(x_1)}{D} - \frac{\phi(x_0)}{D}\right)^{-1},$$

$$T_2 = \frac{1}{2\pi} \sqrt{\phi''(x_2)\phi''(x_0)} \exp\left(\frac{\phi(x_2)}{D} - \frac{\phi(x_0)}{D}\right)^{-1}. \tag{36}$$

Obviously, under the potential $\phi(x_1) = \phi(x_2)$ condition, the lifetime of system in the two symmetric steady states is identical. Figure 2 gives the time of system in steady state under different $a$ and $D$ conditions. Under the circumstance of weak noises, the lifetime of ENSO is about 30-40 months, the probability of the evolution from unstable state to two steady states is almost one-half respectively, thus in a time period $T$ the total time of occurrence of ENSO events is $T_{2\text{theoretical}} = T/2$, and its occurrence probability $\rho_{\text{theoretical}} = 0.5$.

5. Conclusions and discussions

Jin and Tao (1999) used the NCAR monthly averaged sea surface temperature (SST) data from 1961 to 1997, selected the SST means over Nino 3 area (5°S-5°N, 150°-80°W) as signals of the SST changes in the equatorial eastern Pacific, and stipulated that if the amplitude of the SST mean is less than or equal to 0.8 standard deviation ($\pm 0.63°C$), then the SST is in the quasi-normal year phase of ENSO cycle, which is considered here as a steady state $(x_1, y_1)$. According to this stipulation, 1977, 1979, 1980, 1981, 1984, 1990, and 1994 are quasi-normal years $(x_2, y_2)$, and the remains are anomalous climate years, i.e., strong ENSO signals affect the atmospheric general circulation, inducing anomalous climate. El Niño events (the developing phase of ENSO cycles) and La Niña events (the opposite phase of ENSO cycles) in the period from 1961 to 1999 are listed in Tables 1 and 2 (Jin and Tao, 1999), where the ENSO event of 1997 has been added, the time series has been extended to 1999, and therefore $T = 39 \times 12 = 468$ months.

As known from Tables 1 and 2, eight ENSO events (including the developing and opposite phases of ENSO cycles) totally cover a time period of $T_{2\text{real}} = 225$ months.

$$\frac{T_{2\text{real}}}{T} = \frac{225}{39 \times 12} \approx 48.1\%,$$

$$\rho_{\text{ENSO \ real}} = \frac{N_{\text{ENSO}}}{(N_{\text{ENSO}} + N_{\text{QN}})} = \frac{8}{8 + 7} \approx 53.3\%,$$

$$\eta_{\text{real}} = T/(N_{\text{ENSO}} + N_{\text{QN}}) = \frac{39 \times 12}{8 + 7} \approx 32.1 \text{ months}, \tag{37}$$

where $N_{\text{ENSO}}$ and $N_{\text{QN}}$ are ENSO and quasinormal year numbers, respectively. Those are in accord with the theoretical analysis, i.e., the probability of anomalous climate is 48.1%, which possesses 53.3% of the total time aeries length, and its lifetime is about 31.2 months.

Based on the above demonstration and analysis, the following conclusions may be drawn:

(1) The system evolution from an initially unstable state to a steady state has been studied in this paper in terms of $Ω$ FPE expansion theory, and under the dominance of the most unstable manifold the system evolves towards only to a pair of symmetric steady states: one is considered as quasi-normal climatic state, and the other as anomalous climatic state, i.e., strong ENSO state.

(2) This study indicates that even if the initial distribution is not at the origin, known from the deterministic equations the system will evolve towards antisymmetric steady states. However under the effect of stochastic forcing, the system exhibits intrinsic differences from deterministic systems, i.e., when $t \to \infty$, the overwhelming majority probability still distributes half-and-half in a pair of symmetric potential wells, and there is almost no probability distribution in a pair of antisymmetric potential wells. This is in
Table 1. El Niño events

<table>
<thead>
<tr>
<th>Positive SST time</th>
<th>Duration (month)</th>
<th>Extremes time</th>
<th>Extremes</th>
<th>El Niño</th>
<th>Annotation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1963-04–1964-01</td>
<td>10</td>
<td>1963-07</td>
<td>0.75</td>
<td>1963 (weak)</td>
<td>El.f</td>
</tr>
<tr>
<td>1965-04–1966-03</td>
<td>12</td>
<td>1965-12</td>
<td>1.27</td>
<td>1965</td>
<td>El.f</td>
</tr>
<tr>
<td>1972-03–1973-03</td>
<td>13</td>
<td>1972-11</td>
<td>2.01</td>
<td>1972</td>
<td>El.f</td>
</tr>
<tr>
<td>1981-12–1983-09</td>
<td>22</td>
<td>1982-12</td>
<td>3.00</td>
<td>1982/83</td>
<td>El.f / El.s</td>
</tr>
</tbody>
</table>

El.f denotes the first year of El Niño, and El.s the second year of El Niño.

Table 2. La Niña events

<table>
<thead>
<tr>
<th>Negative SST time</th>
<th>Duration (month)</th>
<th>Extremes time</th>
<th>Extremes</th>
<th>La Niña year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1964-02–1965-03</td>
<td>14</td>
<td>1964-05</td>
<td>-1.27</td>
<td>1964</td>
</tr>
<tr>
<td>1974-07–1976-05</td>
<td>23</td>
<td>1975-12</td>
<td>-1.63</td>
<td>1975</td>
</tr>
</tbody>
</table>

accord with observational facts.

(3) Viewed from dynamic macro-theory, quasi-normal climate state and strong ENSO state are two equilibrium states, following a unified dynamic theoretical framework, and have the same temporal/spatial scales; ENSO is a joint name of Southern Oscillations and El Niño cycles and bringing them into a unified framework system in study is reasonable. However viewed from dynamic micro-theory, Southern Oscillations, El Niño, and La Niña reflect different phases of climate, in order to reflect the phase transition between El Niño and La Niña it is necessary to establish their dynamic frameworks respectively.

(4) The real ocean-atmosphere oscillator system is a complex nonlinear system, and discussing only the evolution of unstable states in this paper yields a pair of symmetric potentials, which possess one half of probability respectively. This is in accord with but not completely identical to observations, thus indicating that the system is not thoroughly symmetric potentials. In fact, under $\alpha > 0$ and different condition, the generalized potential function reflects complex and different characteristics (Fig.1c). Studying the simplified ocean-atmosphere coupled model and exploring the intrinsic quality and characteristics of ENSO events are advantageous to the establishment of the interaction theory of ocean-atmosphere.

REFERENCES


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